Announcements

1) HW 5 now due Monday

2) External Reviewers

Would like to meet with math majors from 2:15 - 3 Tuesday 3/12 in CB 2047 (Math Library). There will be cookies!

3) Extra credit due today

Example 1: (matrices)

The <u>elementary matrices</u> in $M_n(\mathbb{R})$ are the collection $\{E_{i,j}\}_{i,j=1}^{n}$ where $E_{i,j}$ is the matrix with a one in the (i,j) position and zeroes in all other positions.

For example, in $M_2(\mathbb{R})$

$$E_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_{2,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = E_{1,2}^{t}$$

$$E_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The set $\{E_{i,j}\}_{i,j=1}^{n}$ is spanning since if

$A = (a_{i,j})_{i,j=1}^{n}$ is in $M_n(\mathbb{R})$,

$$\boxed{A = \sum_{i,j=1}^{n} a_{i,j} E_{i,j}.}$$

For the linear independence, suppose $(a_{i,j})_{i,j=1}^{n}$ are real numbers with

$$\sum_{i,j=1}^{n} a_{i,j} E_{i,j} = O_n \text{ (zero matrix)}$$

This implies $a_{i,j} = 0$ for all i,j since $E_{i,j}$ is the only matrix in the sum with the (i,j) position potentially nonzero.

This reinforces the fact that $\dim(M_n(\mathbb{R})) = n^2$.

Example 2:

$V = \mathbb{R}^5$,

W = all vectors of the form

$$\begin{bmatrix} 10x_1 - 7x_2 \\ -x_2 + 5x_3 - 6x_4 \\ 11x_5 + 121x_2 \\ -8x_4 + x_3 - x_1 \\ 332x_5 \end{bmatrix}$$

Find a basis for W.

Write an element of W as

$$x_1 \overset{v_1}{\begin{bmatrix} 10 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}} + x_2 \overset{v_2}{\begin{bmatrix} -7 \\ -1 \\ 121 \\ 0 \\ 0 \end{bmatrix}} + x_3 \underset{v_3}{\begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + x_4 \underset{v_4}{\begin{bmatrix} 0 \\ -6 \\ 0 \\ -8 \\ 0 \end{bmatrix}} + x_5 \begin{bmatrix} 0 \\ 0 \\ 11 \\ 0 \\ 332 \end{bmatrix} v_5$$

We then see that

$\{v_1, v_2, v_3, v_4, v_5\} = S$

is spanning for W.

Is S linearly independent?

We get v_5 is linearly independent of $T = \{v_1, v_2, v_3, v_4\}$ since v_5 has nonzero 5^{th} coordinate and all the 5^{th} coordinates in T are zero.

We can remove v_2

by the same observation.

We reduce to

$F = \{v_1, v_3, v_4\}$ (third coordinate nonzero)

Then remove v_1

(first coordinate nonzero)

to get

$E = \{v_3, v_4\}$.

But v_3 is not a scalar multiple of v_4, so E is linearly independent.

Therefore, S is a basis for W.

Since S has 5 vectors in it and $W \subseteq \mathbb{R}^5$, $W = \mathbb{R}^5$.

Observations

For a vector space V
of dimension $n < \infty$,
a basis must have
n vectors.

Any subset of V with
less than n vectors is not
spanning for V. Any
subset with more than n
vectors is not linearly
independent.

Definition: (linear transformation)

Let V and W be vector spaces. A linear transformation from V to W is a function $T: V \to W$ satisfying, for all x, y in V and all scalars c,

1) $T(x+y) = T(x) + T(y)$

2) $T(cx) = cT(x)$

Observation: ($\mathbb{R}^n$)

From $\mathbb{R}^n$ to $\mathbb{R}^m$,

the only linear transformations

are $m \times n$ matrices.

How to Find the Matrix of a Linear Transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ (with respect to a given basis)

Given basis $\{b_1, b_2, \ldots, b_n\}$ for $\mathbb{R}^n$, the first column of the matrix is $T(b_1)$.

The second column is $T(b_2)$.

In general, the i^{th} column is $T(b_i)$.

Example 3: (standard basis)

Let $\{e_i\}_{i=1}^{3}$ be the standard basis for $\mathbb{R}^3$.

Suppose

$$T(e_1) = 3e_2 - e_1$$

$$T(e_2) = -6e_3 + 10e_2 + 56e_1$$

$$T(e_3) = 15e_2 + 13e_3$$

$$(T : \mathbb{R}^3 \to \mathbb{R}^3)$$

Find the matrix of T with respect to the standard basis.

$$T(e_1) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 56 \\ 10 \\ -6 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} 0 \\ 15 \\ 13 \end{bmatrix}$$

The matrix is then

$$\begin{bmatrix} -1 & 56 & 0 \\ 3 & 10 & 15 \\ 0 & -6 & 13 \end{bmatrix}$$

Example 4: (nonstandard basis)

Let $b_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

$b_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Then $\{b_1, b_2\}$ is a basis for $\mathbb{R}^2$. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(b_1) = -6\, b_2$$

$$T(b_2) = 3b_1 + 13\, b_2$$

Find the matrix of T with respect to this basis.

We can write, in terms of $\{b_1, b_2\}$,

$$T(b_1) = \begin{array}{cl} \left[\begin{array}{c} 0 \\ -6 \end{array}\right] & \begin{array}{l} \leftarrow b_1 \text{ coordinate} \\ \leftarrow b_2 \text{ coordinate} \end{array} \end{array}$$

$$T(b_2) = \begin{array}{cl} \left[\begin{array}{c} 3 \\ 13 \end{array}\right] & \begin{array}{l} \leftarrow b_1 \text{ coordinate} \\ \leftarrow b_2 \text{ coordinate} \end{array} \end{array}$$

The matrix is

$$\begin{bmatrix} 0 & 3 \\ -6 & 13 \end{bmatrix}$$

Back to Standard

What is the matrix of T with respect to the standard basis?

You need to find numbers a_1, a_2, a_3, a_4 with

$$a_1 b_1 + a_2 b_2 = e_1$$

$$a_3 b_1 + a_4 b_2 = e_2$$

to figure out what T does to e_1 and e_2.

Solve

$$a_1 \underbrace{\begin{bmatrix} 5 \\ -2 \end{bmatrix}}_{b_1} + a_2 \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{b_2} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}$$

This becomes the matrix equation

$$\underbrace{\begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}}_{S} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\det(S) = 15 - 2 = 13$, so S is invertible

$$S^{-1} = \frac{1}{13}\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}.$$

Applying S^{-1} to

$$S\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix},$$

we get

$$\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \overset{\frac{1}{13}\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}}{\overset{\shortparallel}{S^{-1}}}\begin{bmatrix}1\\ 0\end{bmatrix},$$

so $a_1 = \frac{3}{13}$, $a_2 = \frac{2}{13}$.

Similarly, applying

S^{-1} to

$$S\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

gives

$$a_3 = \frac{1}{13}, \quad a_4 = \frac{5}{13}$$

So

$$T(e_1) = T\left(\frac{3}{13} b_1 + \frac{2}{13} b_2\right)$$

$$= \underbrace{\frac{3}{13} T(b_1) + \frac{2}{13} T(b_2)}_{\text{linearity of } T}$$

$$= \frac{3}{13}(-6 b_2) + \frac{2}{13}(3b_1 + 13 b_2)$$

$$= \frac{6}{13} b_1 + \frac{8}{13} b_2$$

$$= \frac{6}{13}\begin{bmatrix} 5 \\ -2 \end{bmatrix} + \frac{8}{13}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{22}{13} \\ \frac{12}{13} \end{bmatrix}$$

and

$$T(e_2) = T\left(\frac{1}{13} b_1 + \frac{5}{13} b_2\right)$$

$$= \frac{1}{13} T(b_1) + \frac{5}{13} T(b_2)$$

$$= \frac{1}{13}(-6b_2) + \frac{5}{13}(3b_1 + 13b_2)$$

$$= \frac{15}{13} b_1 + \frac{59}{13} b_2$$

$$= \frac{15}{13}\begin{bmatrix} 5 \\ -2 \end{bmatrix} + \frac{59}{13}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{13} \\ \frac{147}{13} \end{bmatrix}$$

Then the matrix of T in the standard basis is

$$\boxed{\frac{1}{13}\begin{bmatrix} 22 & 16 \\ 12 & 147 \end{bmatrix}}$$

Observe that

$$STS^{-1}$$

$$= \frac{1}{13}\begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 3 \\ -6 & 13 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}}_{\parallel}$$

$$= \frac{1}{13}\begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 & 15 \\ 8 & 59 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 22 & 16 \\ 12 & 147 \end{bmatrix}$$

This is not a coincidence!